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Vorticity and curvature at a free surface

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For two-dimensional flow there is a simple relation between the vorticity at a stress-free surface and the surface curvature. In this note the relation is generalized to flow in three dimensions. It is shown that in addition to a component of vorticity perpendicular to the flow there is also a component parallel to the direction of flow. The latter vanishes only at an umbilical point or when the flow is in one of the two principal directions of curvature.

1. Introduction

The boundary conditions prevailing at a free surface where both tangential components of the stress tensor vanish is of considerable interest for studies of surface waves, rising bubbles, free-surface turbulence, and other applications. In purely two-dimensional flow, there exists a simple relation

$$\omega = -2\kappa q \tag{1}$$

between the vorticity ω, the tangential velocity q and the curvature κ, assuming the surface to be steady (see Longuet-Higgins 1953, 1992; Batchelor 1967). The question we consider here is how to find an extension of the formula to general, steady surfaces in three dimensions.

Just this problem has in fact been treated by Batchelor in his textbook (1967, §5.14) by the use of orthogonal curvilinear coordinates. But although such coordinates can always be chosen in general, such a choice necessarily imposes some restrictions on the results as there stated. It turns out, as we shall show, that in addition to a component of vorticity perpendicular to the flow, which is given by an expression similar to equation (1), there is in general a component of vorticity parallel to the flow, which vanishes only under special conditions; see equation (18). An alternative derivation is given in §3.

Other authors (Creswell & Morton 1995; Rood 1995; Wu 1955; Sarpkaya 1996) have considered the same problem in special cases or with generalized notation which tends to hinder a physical interpretation of the results, and to conceal their essential simplicity. An expression equivalent to equation (18) has been obtained by Peck & Sigurdson (1997), but the derivations given here are believed to be more elementary.

2. Proof A

In the following we shall adopt the same approach as Batchelor (1967) but with a slightly modified notation.

Let us take orthogonal curvilinear coordinates (ξ_1, ξ_2, ξ_3) so that ξ_3 is in the direction of the outward normal to the surface. The elements of length in the three coordinate directions are denoted by $h_1\,d\xi_1, h_2\,d\xi_2, h_3\,d\xi_3$ respectively, and the three components of velocity by u_1, u_2, u_3. Then the component of vorticity in the ξ_1-direction is given by

$$\omega_1 = \frac{1}{h_2 h_3}\left[\frac{\partial(u_3 h_3)}{\partial\xi_2} - \frac{\partial(u_2 h_2)}{\partial\xi_3}\right] \tag{2}$$

(see Batchelor 1967, Appendix 2), and the component e_{32} of the rate-of-strain tensor is given by

$$2e_{32} = \frac{1}{h_2 h_3}\left[h_3^2\frac{\partial(u_3/h_3)}{\partial\xi_2} + h_2^2\frac{\partial(u_2/h_2)}{\partial\xi_3}\right]. \tag{3}$$

Equations (2) and (3) lead immediately to the identity

$$\omega_1 + 2e_{32} = \frac{2}{h_2 h_3}\left(h_3\frac{\partial u_3}{\partial\xi_2} - u_2\frac{\partial h_2}{\partial\xi_3}\right). \tag{4}$$

If now the surface is steady and the velocity is measured relative to the surface we must have

$$u_3 \equiv 0, \quad \frac{\partial u_3}{\partial\xi_2} = 0 \tag{5}$$

and if moreover the components of tangential stress both vanish at the surface then

$$e_{31} = e_{32} = 0. \tag{6}$$

Hence equation (4) reduces to

$$\omega_1 = -\frac{2u_2}{h_2 h_3}\frac{\partial h_2}{\partial\xi_3} \tag{7}$$

and similarly we find

$$\omega_2 = \frac{2u_1}{h_1 h_3}\frac{\partial h_1}{\partial\xi_3}. \tag{8}$$

Because the coordinates are mutually orthogonal, the normal curvature κ_1 of the ξ_1-axis is given by

$$\kappa_1 = -\frac{1}{h_1}\frac{\partial h_1}{\partial n}, \tag{9}$$

where n is distance measured normal to the surface (κ_1 is taken to be positive if the surface curves outwards). Since $dn = h_3\,d\xi_3$ we have

$$\kappa_1 = -\frac{1}{h_1 h_3}\frac{\partial h_1}{\partial\xi_3} \tag{10}$$

so that equation (5) becomes

$$\omega_1 = 2\kappa_2 u_2. \tag{11}$$

Similarly we find

$$\omega_2 = -2\kappa_1 u_1. \tag{12}$$

There is no restriction on the normal component of vorticity ω_3.

At first sight it might appear that the above formulae could be applied to any system of orthogonal coordinates on the surface $\xi_3 = 0$; in other words, that the vorticity vector was always normal to the velocity vector (Batchelor 1967). But this is not the case. The assumption that the coordinate system is *triply* orthogonal implies that the choice of coordinates (ξ_1, ξ_2) is more special, as is shown by Dupin's theorem (see Weatherburn 1961, p. 211):

The curves of intersection of the surfaces of a triply orthogonal system are lines of curvature on each.

Hence for equations (11) and (12) to be generally valid, the coordinate curves ξ_1 and ξ_2, must be *lines of curvature* on the surface $\xi_3 = 0$. That is to say, the normal to each coordinate curve is coplanar with the normal to the surface at the origin. Moreover κ_1 and κ_2 must be the two principal curvatures, κ_a and κ_b, say.

In the case when the flow velocity q is in one of the principal directions, say along the ξ_1-axis, then $u_2 = 0$ and from equations (7) and (8) we have

$$\omega_1 = 0, \quad \omega_2 = -2\kappa_1 u_1. \tag{13}$$

A physical interpretation is as follows. To second order, the surface has locally the form of an ellipsoid or hyperboloid with reflexive symmetry about the ξ_1-axis. This symmetry means that ω_1 must vanish, while ω_2 is given by an expression analogous to equation (1) for the two-dimensional situation.

On the other hand suppose that the velocity q is in a direction making some general angle α with the axis of ξ_1, that is

$$u_1 = q\cos\alpha, \quad u_2 = q\sin\alpha. \tag{14}$$

The two components $\omega_\parallel$ and $\omega_\perp$ of the surface vorticity parallel and perpendicular to the flow are given by

$$\left.\begin{aligned} \omega_\parallel &= \omega_1\cos\alpha + \omega_2\sin\alpha, \\ \omega_\perp &= -\omega_1\sin\alpha + \omega_2\cos\alpha. \end{aligned}\right\} \tag{15}$$

Substituting from equations (11), (12) and (14) we obtain

$$\omega_\perp = -2q(\kappa_a\cos^2\alpha + \kappa_b\sin^2\alpha). \tag{16}$$

By Euler's theorem (Weatherburn 1961, p. 73) this can be written

$$\omega_\perp = -2q\kappa_n \tag{17}$$

where κ_n denotes the normal curvature of the particle path.

But from (15) we have also

$$\omega_\parallel = 2q(\kappa_b - \kappa_a)\cos\alpha\sin\alpha \tag{18}$$

showing that the component of vorticity parallel to the flow does not vanish in general. Exceptions may occur when $\alpha = 0$ or $\alpha = \frac{1}{2}\pi$ (when the flow is parallel to one of the principal directions) or when $\kappa_a = \kappa_b$, that is if the two principal curvatures are equal so the point on the surface is an umbilic. Physically, the non-vanishing vorticity (18) may be thought of as a twisting motion induced by the twisting of the normal $\boldsymbol{n}$ along the path of the particle.

3. Proof B

In view of the vanishing of e_{13} and e_{23}, the rate-of-strain tensor $\boldsymbol{e}$ has the form

$$\boldsymbol{e} = \begin{pmatrix} e_{11} & e_{12} & 0 \\ e_{21} & e_{22} & 0 \\ 0 & 0 & e_{33} \end{pmatrix}. \tag{19}$$

One of the eigenvalues of this tensor is clearly e_{33}, and the corresponding principal direction is in the direction of the normal. In other words, *one of the principal axes of strain is always normal to the surface*. But by the well-known analysis of strain given for example in Batchelor (1967, §2.3), any line of fluid particles parallel to a principal axis of strain must be in rotation with angular velocity $\frac{1}{2}\boldsymbol{\omega}$, where $\boldsymbol{\omega}$ is the vorticity. Since one axis of strain is always normal to the free surface, the normal is a kind of continuation of such a line of particles. It follows that the normal must rotate with angular velocity $\frac{1}{2}\boldsymbol{\omega}$.†

From the above remark we are led directly to an alternative proof of our results. For, since the unit normal $\boldsymbol{n}$ is a vector of constant length rotating with the same angular velocity $\frac{1}{2}\boldsymbol{\omega}$ as the principal axes of strain, we have

$$\frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}t} = \frac{1}{2}\boldsymbol{\omega} \wedge \boldsymbol{n}. \tag{20}$$

Writing $\mathrm{d}s$ for the element of path length and $q = \mathrm{d}s/\mathrm{d}t$ we obtain

$$\boldsymbol{\omega} \wedge \boldsymbol{n} = 2q\frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}s} \tag{21}$$

which is equivalent to equations (11) and (12) above. For, taking the vector product of $\boldsymbol{n}$ with each side of (21) we have for the tangential part of the vorticity

$$\boldsymbol{\omega}_s = \boldsymbol{\omega} - (\boldsymbol{n}\cdot\boldsymbol{\omega})\,\boldsymbol{n} = 2q\boldsymbol{n} \wedge \frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}s}. \tag{22}$$

If (ξ_1, ξ_2) are general curvilinear coordinates on the surface, not necessarily orthogonal, $\boldsymbol{r}$ is the position vector and r_i denotes $\partial r/\partial \xi_i$ it follows that

$$\boldsymbol{\omega}_s \cdot \boldsymbol{r}_i = 2q\left[\boldsymbol{n}, \frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}s}, \boldsymbol{r}_i\right], \quad i = 1, 2. \tag{23}$$

(The square brackets denote triple vector products.) But

$$\mathrm{d}\boldsymbol{n} = \boldsymbol{n}_1\,\mathrm{d}\xi_1 + \boldsymbol{n}_2\,\mathrm{d}\xi_2 \tag{24}$$

where $\boldsymbol{n}_i$ denotes $\partial\boldsymbol{n}/\partial\xi_i$. If now the coordinates x_1 and ξ_2 are mutually orthogonal, then

$$h_1\frac{\mathrm{d}\xi_1}{\mathrm{d}s} = \cos\alpha, \quad h_2\frac{\mathrm{d}\xi_2}{\mathrm{d}s} = \sin\alpha. \tag{25}$$

Hence

$$\boldsymbol{\omega}_s \cdot \boldsymbol{r}_i = 2q\{(1/h_1)\,[\boldsymbol{n}, \boldsymbol{n}_1, \boldsymbol{r}_i]\cos\alpha + (1/h_2)\,[\boldsymbol{n}, \boldsymbol{n}_2, \boldsymbol{r}_i]\sin\alpha\}. \tag{26}$$

But the triple products in equation (26) can be expressed in terms of the fundamental magnitudes at a point on the surface (see the Appendix) and in particular when the coordinates ξ_1, ξ_2 are in the principal directions of curvature, then we find

$$\left.\begin{aligned} &[\boldsymbol{n}, \boldsymbol{n}_1, \boldsymbol{r}_1] = 0, && [\boldsymbol{n}, \boldsymbol{n}_1, \boldsymbol{r}_2] = -h_1 h_2 \kappa_1, \\ &[\boldsymbol{n}, \boldsymbol{n}_2, \boldsymbol{r}_1] = h_1 h_2 \kappa_2, && h_2[\boldsymbol{n}, \boldsymbol{n}_2, \boldsymbol{r}_2] = 0. \end{aligned}\right\} \tag{27}$$

Thus equations (26) reduce to (11) and (12). The rest of §2 then follows.

Appendix

For convenience and possible use in applications we state here some known formulae for the triple vector products in $[\boldsymbol{n}, \boldsymbol{n}_j, \boldsymbol{r}_i]$ occurring in equation (26).

† The fluid is not of course in a solid-body rotation, as was inadvertently stated by Longuet-Higgins (1992) in the two-dimensional case. That would be true only if $\boldsymbol{e}$ were to vanish identically. At the same time as rotating, the elements of fluid are also undergoing an arbitrary straining motion.

In any system of curvilinear coordinates (ξ_1, ξ_2) on a surface in three dimensions we can define the first-order magnitudes

$$E = \boldsymbol{r}_1 \cdot \boldsymbol{r}_1, \quad F = \boldsymbol{r}_1 \cdot \boldsymbol{r}_2, \quad G = \boldsymbol{r}_2 \cdot \boldsymbol{r}_2, \tag{A 1}$$

where $\boldsymbol{r}$ is the position vector and suffices i denote differentiation with respect to ξ_i. These quantities occur naturally in the general expression for the arclength $\mathrm{d}s$:

$$\mathrm{d}s^2 = E\,\mathrm{d}\xi_1^2 + 2F\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 + G\,\mathrm{d}\xi_2^2. \tag{A 2}$$

The second-order magnitudes L, M, N, are defined by

$$L = \boldsymbol{n} \cdot \boldsymbol{r}_{11}, \quad M = \boldsymbol{n} \cdot \boldsymbol{r}_{12}, \quad N = \boldsymbol{n} \cdot \boldsymbol{r}_{22}. \tag{A 3}$$

Then it may be shown (Weatherburn 1961, §27, for example) that

$$\left.\begin{aligned} H[\boldsymbol{n}, \boldsymbol{n}_1, \boldsymbol{r}_1] &= EM - FL, \quad H[\boldsymbol{n}, \boldsymbol{n}_1, \boldsymbol{r}_2] = FM - GL, \\ H[\boldsymbol{n}, \boldsymbol{n}_2, \boldsymbol{r}_1] &= EN - FM, \quad H[\boldsymbol{n}, \boldsymbol{n}_2, \boldsymbol{r}_2] = FN - GM, \end{aligned}\right\} \tag{A 4}$$

where

$$H^2 = EG - F^2. \tag{A 5}$$

If the coordinates are orthogonal, then $F = 0$, $E = h_1^2$ and $G = h_2^2$, and

$$\left.\begin{aligned} H[\boldsymbol{n}, \boldsymbol{n}_1, \boldsymbol{r}_1] &= EM, \quad H[\boldsymbol{n}, \boldsymbol{n}_1, \boldsymbol{r}_2] = -GL, \\ H[\boldsymbol{n}, \boldsymbol{n}_2, \boldsymbol{r}_1] &= EN, \quad H[\boldsymbol{n}, \boldsymbol{n}_2, \boldsymbol{r}_2] = -GM. \end{aligned}\right\} \tag{A 6}$$

If further the coordinate curves are in the two principal directions then $M = 0$, $L/E = \kappa_a$, $N/G = \kappa_b$ and we obtain equations (27).

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